*International Journal of Theoretical Physics*, Vol. 12, No. 6 (1975), pp. 401–406

# Mach's Principle in Cosmology

E. IHRIG

*Department of Mathematics, University of New Brunswick, Fredericton, New Brunswick, Canada E3B 5A3*

*Received: 28 August 1974*

*Abstract*

In a certain class of cosmological models it is shown that every Ricci flat complete space-time must be locally flat.

## 1. *Introduction*

Recently space-times have been found that violate Mach's principle (Abe, 1971; Ozsvath & Schuching, 1962). These space-times are complete and empty, but not locally flat (i.e. they have no matter and a non-vanishing gravitational field). This is just another expression of the fact that the field equations do not completely determine the gravitational field. Einstein (1917) himself recognized this problem and attempted to solve it by illuminating the need for boundary conditions using a universe with compact spatial section. Unfortunately, boundary conditions are still needed unless the whole space-time is compact. Compact space-times are not interesting in themselves since they are acausal. However, those space-times that are covering spaces of compact space-times need not be acausal and require no boundary conditions just as the compact space-times. Thus one would hope that this aspect of Mach's principle would apply if one restricted oneself to these space-times which are called periodic (see Ihrig & Sen, 1974). It is the purpose of this note to give a result in this direction.

Unfortunately this result is only partial since it uses other restrictions than just periodicity. We have neither been able to prove every Ricci flat periodic space-time is flat nor produce a counter example. However, we suspect that there is no counter example since if Bochner theory held for pseudo-Riemannian metrics then the proof of 2.4 would show this. Although this more general question is left open, our partial result is of interest since the most confining restriction we use from a physical point of view is the periodicity condition.

Copyright © 1975 Plenum Publishing Corporation. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, microfilming, recording, or otherwise, without written permission of the publisher.

## 2. *Mach's Principle*

To satisfy Mach's principle we would like to find a class $\mathscr{C}$ of space-times which have the following properties:

(i) if $M \in \mathscr{C}$ and if $M$ is empty it is locally flat,
(ii) every $M \in \mathscr{C}$ is complete,
(iii) $\mathscr{C}$ contains models which agree with observations as well as any Robertson–Walker metric (with $k = 0$).

Since $\mathscr{C}$ will be taken to be a subclass of periodic space-times we will need to include the cosmological constant in Einstein's equations:

$$R_\mu^{\ \nu} - \tfrac{1}{2} R g_\mu^{\ \nu} + \Lambda g_\mu^{\ \nu} = -\frac{\sigma \pi K}{c^2} T_\mu^{\ \nu}$$

We will include a fourth criterion for our class that the cosmological constant is not an independent constant. Using the fact that the only constant that can be constructed from the geometry is the integral of a scalar and that this integral must be of the same dimension as $R$ we make the following assumption.

2.1. *Assumption.* Let $M$ be a periodic space-time which is the covering space of $\bar{M}$ then

$$\Lambda = l \int_{\bar{M}} R(\sqrt{-g})\, dx^1\, dx^2\, dx^3 \ldots dx^n \Bigg/ \int_{M'} \frac{dx^1\, dx^2\, dx^3\, dx^n}{\sqrt{-g}}$$

where $l$ is a fixed dimensionless constant $l \neq (\frac{1}{2}n - 1)/n = n_0$, $n = \dim M$. Now we are able to define $\mathscr{C}$.

2.2. *Definition.* $M \in \mathscr{C}$ iff

(a) $M$ is periodic with $\bar{M}$ its corresponding compact space-time,
(b) $\bar{M}$ admits $R^{n-1}$ as an isometry group which generates a $n-1$-dimensional space-like distribution,
(c) $M$ is finely causal ($\bar{M}$ has no trapped null curves (see Ihrig & Sen, 1975).

Condition (b) is needed in the proof of our following theorem, but it seems that the theorem is probably true without this assumption for $n = 4$. Since (b) does not interfere with (iii) of our conditions on $\mathscr{C}$, this is not too much of a problem. (c) is a very weak condition that assures that periodic space-times are complete (Ihrig & Sen, 1975). (Note that not every periodic space-time is complete (Misner, 1963).) So the major assumption on $\mathscr{C}$ is (a).

We now verify that $\mathscr{C}$ has the desired properties. (iii) follows from noting that any Robertson-Walker metric is the same as a periodic Robertson-Walker metric for as far as observations can be made (unless a big bang has been observed). The $k = 0$ metrics satisfy (b) and all the periodic metrics satisfy (c).

Concerning (ii), (c) assures all the space-times in $\mathscr{C}$ are complete.

We must only deal with (i). We take care of this condition with the following theorem:

2.3. *Theorem.* Suppose $M \in \mathscr{C}$ of (2.2). Then if $M$ is empty ($T_{ij} = 0$) then $M$ is locally flat.

*Proof.* Assume $T_{ij} = 0$. We first show $R_{ij} = 0$. Using Einstein's equations we find

$$R = (1/n_0)\Lambda \qquad \text{where } n_0 = (\tfrac{1}{2}n - 1)/n$$

Thus, using 2.1, if $\omega = (\sqrt{-g})\, dx^1\, dx^2\, dx^3 \ldots dx^n$ then

$$\Lambda = l \int_{\bar{M}} \left(\frac{1}{n_0}\Lambda\right)\omega \Big/ \int_{\bar{M}} \omega$$
$$= \frac{l}{n_0}\Lambda$$

Since $l \neq n_0$ we have $\Lambda = 0$. Thus $R = \Lambda = 0$ and Einstein's equations gives $R_{ij} = 0$.

We will now continue working only with $\bar{M}$. $\bar{M}$ is Ricci flat and compact from the above considerations. Also $\bar{M}$ may be assumed to be orientable for otherwise one may replace $\bar{M}$ by the appropriate 2-fold covering. Let $D$ be the distribution generated by $R^{n-1}$ in 2.2(b). Let $v_1, \ldots, v_{n-1}$ be the Killing vector fields generated by $l_1, \ldots, l_{n-1} \in R^{n-1}$, $\{l_i\}$ any bases of $R^{n-1}$. Since $\bar{M}$ is a compact manifold we have for any $i$ that

$$0 = \int_{\bar{M}} (\mathrm{Ric}\,(v_i, v_i) + \mathrm{trace}\, A_{v_i}A_{v_i})\omega$$

But $\mathrm{Ric}\,(v_i, v_i) = 0$. So that the only term left is trace $A_{v_i}A_{v_i}$. We have

$$\mathrm{trace}\,(A_{v_i}A_{v_i}) = -\sum_e \epsilon(e)(A_{v_i}X_e, A_{v_i}X_e)$$

where $(X_0, D) = 0$, $(x_0, x_0) = -1$, $x_e$ is an orthonormal basis of the tangent space at a given point and $\epsilon(e) = -1$ if $e = 0$, $\epsilon(e) = 1$ if $e \neq 0$. If we show that

$$\epsilon(e)(A_{v_i}(X_e), A_{v_i}(X_e))$$

is less than zero for every $e$ then we may conclude that

$$(A_{v_i}X_e, A_{v_i}X_e) = 0 \;\forall\, e$$

If also each $A_{v_i}X_e$ is either time-like or space-like then

$$A_{v_i}X_e = 0 \;\forall\, e$$

and thus $v_i$ is totally geodesic since $A_{v_i}X_e = -D_{X_e}v_i = 0$.

Let us show these two things. We will in fact show $A_{v_i}X_0$ is space-like and $A_{v_i}X_e$ is time-like for $e \neq 0$, which will give us both of our requirements. First we deal with $A_{v_i}x_0$.

$$(A_{v_i}x_0, x_0) = -(A_{v_i}x_0, x_0)$$

which gives $(A_{v_i}x_0, x_0) = 0$ or $A_{v_i}x_0$ is space-like. Now we need only to show $A_{v_i}x_e$ are time-like. We will show

$$(A_{v_i}v_j, v_k) = 0 \qquad j, k \neq 0$$

Thus $A_{v_i}v_j = \lambda_j v_0$ and so $A_{v_i}x_e$ will be a constant times $v_0$ since $x_e$ is a linear combination of the $v_j$. Now since $[v_i, v_j] = 0$ we have

$$(A_{v_i}v_j, v_k) = (A_{v_j}v_i, v_k) = -(v_i, A_{v_j}v_k)$$

Also since $(A_{v_i}v_j, v_k) = -(A_{v_i}v_k, v_j)$ we have

$$(A_{v_i}v_j, v_k) = (v_i, A_{v_k}v_j) = (v_i, A_{v_j}v_k)$$

Thus $(A_{v_i}v_j, v_k)$ is zero. This establishes that $v_i$ is totally geodesic. So $\bar{M}$ must be locally flat and thus $M$ is also locally flat as desired.

We have shown $\mathscr{C}$ is a collection of cosmological models which satisfies the desired features stated above. An interesting question is whether $\mathscr{C}$ can be extended to include more periodic space-times. One would suspect so because of the following observation:

2.4. *Remark.* Suppose $M$ is a Riemannian metric, is periodic, and is four-dimensional. Suppose also that $\bar{M}$ admits some space-time structure. Then if $M$ is Ricci flat it is flat.

*Proof.* Suppose $M$ is Ricci flat. Then so is $\bar{M}$, the compact metric associated with $M$. $\bar{M}$ admits a space-time structure and dim $(\bar{M}) = 4$ means $H^1(\bar{M}, R) \neq 0$ (see Ihrig & Sen, 1974). Thus $\bar{M}$ must have a harmonic vector field $v_1$. According to Bochner (1974), $v_1$ is totally geodesic. Thus locally $\bar{M} = R \times M_3$ where $M_3$ is a 3-dim Ricci flat manifold. But every 3-dim Ricci flat manifold is locally flat. Thus $\bar{M}$ is locally flat and so is $M$ (see Fisher & Wolf, 1974, for a different approach to the last two lines of this proof).

If we were to try to apply this remark to the Lorentz case we would need either one time-like Killing vector field or three space-like ones (see Komar, 1966). The assumption of a time-like Killing vector field in cosmology is generally too strong to be of interest; thus one would have to assume the existence of three space-like hypersurface orthogonal Killing vector fields to proceed. This is reasonable, but one can give a more direct proof in an arbitrary dimensional setting if one assumes this much, as we did in 2.3.

## *References*

Abe, S. (1971). The space-time with an anti-mach and $V_4(\pi)$, *Tensor*, **22**, 261.

Bochner, S. (1946). Vector fields and Ricci curvature, *Bulletin of the American Mathematical Society*, **52**, 776.

Einstein, A. (1917). Kosmologische Betrachtungen zur Allgemeinen Relativitätstheorie, *Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin.*

Fisher, A. and Wolf, J. (1974). The Calabi construction for compact Ricci flat Riemannian manifolds, *Bulletin of the American Mathematical Society*, **80**, 92.

Ihrig, E. and Sen, D. K. (1974). A class of singular spacetimes, *General Relativity and Gravitation*, **5** (5), 593.

Ihrig, E. and Sen, D. K. (1975). Analytic singularities and geodesic completeness II, to appear.

Komar, A. (1966). Analytic continuation and global theorems in general relativity, article in *Perspectives in Geometry and Relativity*. Indiana University.

Misner, C. (1963). The flatter regions of Newman, Unti, and Tamburino's generalized Schwarzchild space, *Journal of Mathematical Physics*, **4**, 924.

Ozsvath, T. and Schuching E. (1962). Article in *Recent Developments in General Relativity*, p. 339. Pergamon.